drain bias and this is well matched by the model in general, although the agreement is poorer for the 1.0 V drain bias. Graphs comparing measured and simulated S-parameters as a function of gate bias show good agreement. The fit to the I_{ds} data is good in general except near pinch-off where the independence of λ, $m\lambda$, κ and $m\kappa$ from V_{gs} may be a source of some model inaccuracy.

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Nonlinear Mixer Gain Calculations for Josephson Junctions

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***Abstract*—We have numerically solved the steady-state solutions of the initial value problem associated with a current-driven Josephson weak-link junction shunted by an Ohmic resistance. The nonlinear mixing action of the junction leads to Shapiro steps in the dc response with step height in units of the mixing frequency. Mixer gains have been calculated with a wide range of parameter values and intrinsic chaos are observed whenever Shapiro steps are prevalent.**

I. Introduction

The current-driven Josephson weak-link can be formulated in terms of the resistively shunted junction (RSJ) model that may be cast in the form of a first-order differential equation, shown in (1). In integrating the equation from $t = 0$ to obtain a steady-state solution, one is faced with the problem that the initial phase, $\phi(0)$, across the junction is unknown. The solution to (1) is very sensitive to the initial condition on ϕ, and a slight change in the initial condition may result in chaotic behavior of ϕ [1]. In the absence of noise the steady-state solution, if it exists, the system is required to return to its initial phase of 2π after one period of the sinusoidal drive (or drives). This determines the asymptotic solution of the system and results in Shapiro steps in units of the mixing frequency in the dc response. We found that intrinsic chaos is most likely to be observed near these step edges. In the presence of noise, the system may not be able to return to its initial phase after one period, and hence no steady-state solution is possible. This leads to extrinsic chaos, since it can be induced by external noise [2]. Traditionally, the Josephson mixer gain is calculated from a linearized perturbation theory [3]–[7]. In the linearized solution, one assumes that the response in ϕ can be approximated by a linear combination of the dc and rf responses, since the rf excitation amplitude is small compared to the dc biasing current. As such, one can then apply circuit theory analysis as in the so-called "conversion matrix method.&r Here we present a systematic method by which the gain is calculated in the nonlinear regime. We find that the mixer gain can be much greater than unity for small carrier current and large local oscillator current under optimal coupling of the load resistance to the junction-shunting circuit. This high gain effect of a Josephson mixer has also been observed experimentally [8].

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II. Calculation

The Josephson mixer circuit shown in the inset of Fig. 3 leads to the following first-order equation [6]:

$$i_D + i_S \cos\omega_S t + i_L \cos\omega_L t + i_N(t) - \sin\phi = \tau(d\phi/dt), \quad (1)$$

where ϕ is the phase difference of the superconducting wave function across the junction and i_D, i_S, i_L, and i_N are, respectively, the dc current, rf currents at the signal carrier frequency, ω_S, the local-oscillator frequency, ω_L, and the noise current, all normalized to the critical current of the junction, I_c. In (1), $\tau = \hbar/2eR_G I_c$, $R_G^{-1} = Z_G^{-1} + R^{-1}$ and R and Z_G are the shunting and external load resistances. We assume the thermal current i_N possesses normal distribution with $\langle i_N\rangle = 0$ and $\langle i_N^2\rangle = 4k_BTB/RI_c^2$, where B denotes the frequency bandwidth of the detector and T is the junction temperature. Note that in (1) we have ignored the shunting capacitance across a Josephson weak-link junction, since it is assumed to be small. Therefore, if the initial value of ϕ is known at $t = 0$, denoted as α, $\phi(\alpha;\, t)$, can be calculated by integrating (1) utilizing a fourth-order Runge-Kutta algorithm in double-precision arithmetic.

If there are no rf currents and only a dc current applied to the junction, the solution is straightforward, sınce the differential flux across the junction is zero, and hence $\alpha = \sin^{-1}(i_D/I_c)$. In a typical mixer experiment, two rf currents and one dc current source are applied to the junction simultaneously. If the dc source is applied firstly, ϕ is unknown upon application of the rf currents. The onset times of these sources may not be precisely noticed under most experimental conditions. Similarly, if we reverse the experiment, ϕ is still an unknown quantity in the presence of all three sources. The dilemma is then what to choose for an initial condition on ϕ in order to uniquely solve for ϕ as a function of time. In this paper, we are interested ın the steady-state solution after all the transients have died out. One can solve this problem empirically by choosing different α values and noting what is the response of ϕ after one common period T, $\phi(\alpha;\, T)$. Here, we assume T to be the period at the mixing frequency, which should not be confused with the symbol used for temperature. If one plots the phase change, $\phi(\alpha;\, T)$-α, as a function of α, one may find a value of α, say α_o, at which the phase change is

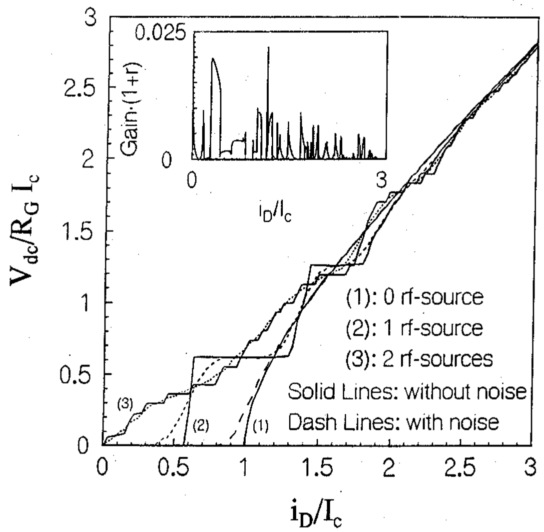

Fig. 1. 1-dc response of a Josephson weak-link junction.

zero modules by 2π. We choose α_o as the initial phase of the system, since it does not change with time. As such, a steady-state solution of (1) is obtained. We found that α_o is unique for the following calculations (that is, the steady-state solution of (1) is unique in the absence of noise): In the presence of noise, we integrate (1) from $t = 0$ to $t = T$, still using the same initial conditions $\phi(0) = \alpha_o$. However, the noise current $i_N(t)$ will then be randomly generated through a random number generator. The solutions are calculated one hundred times with a different random noise generation on each of the runs. We average over these one hundred solutions to arrive at the average solution of $\phi(t)$ that satisfies (1) in the presence of noise. The above solution procedure is roughly valid if the noise level is not too high. If the noise level is high, the steady-state solution of the system may no longer exist and extrinsic chaotic motio edges.

III. Results

In the following, the voltage will be normalized with respect to $R_G I_c$. The induced voltage at the Josephson junction is

$$V = \tau \, d\phi/dt. \tag{2}$$

The dc and rf responses of the system may be expressed in the frequency domain by Fourier-transforming the time-domain solutions. The resultant dc and mixing voltages are denoted as V_{dc} and V_m, which are evaluated at dc and mixing frequency, $\omega_M = \omega_S - \omega_L$, respectively. The mixer gain is defined as

$$G = |V_M^2/i_S V_S|/(r+1), \tag{3}$$

where $r = Z_G/R$ and V_S denotes the voltage at the carrier frequency, ω_S. In the following, we assume $f_S = \omega_S/2\pi = 10GHz$, $f_L = \omega_L/2\pi = 9$ GHz, and $T = 1$ nsec. Fig. 1 shows the dc response of the system if the junction is excited by 0, 1, and 2 rf sources, as referenced in curves (1), (2), and (3), respectively. In Fig. 1, $\tau = 0.01$ nsec and $\langle i_N^2 \rangle = 0$ for the solid lines and $\langle i_N^2 \rangle = 0.25$ for the dash lines, which have been averaged over 100 runs. For curves (1), we apply one source across the junction, $i_D = 1$, $i_S = i_L = 0$, which reduces to the well-known V-I characteristics of a Josephson junction with and without noise. For curves (2), we apply two sources in which $i_D = i_S = 1$, $i_L = 0$. Shapiro steps of height $\omega_S \tau$ (= 0.62831) are clearly calculable. For curves (3) in which we have three sources $i_D = i_S = i_L = 1$, the Shapiro steps are composed of smaller steps in units of ω_M, which is one-tenth of the height observed in curves (2). We note that the noise effect in Fig. 1 is to round off the sharp

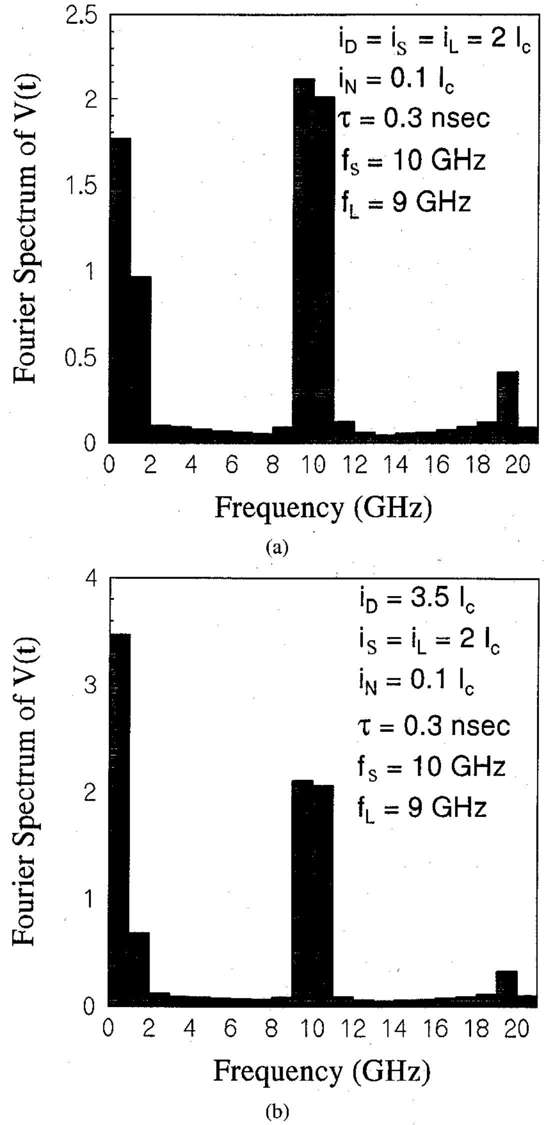

Fig. 2. Fourier spectrum of $V(t)$ at two i_D va;ies. (a) $i_D = 2I_c$. (b) $i_D = 3.5I_c$.

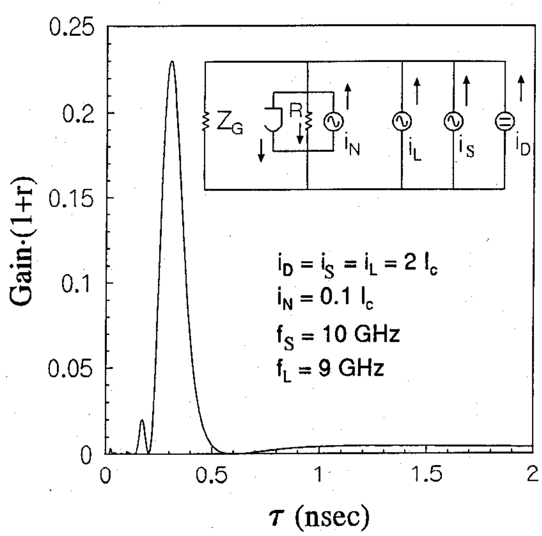

Fig. 3. Mixer gain as a function of τ.

corners appearing in the noiseless cases, as expected. The inset of Fig. 1 shows the mixer gain associated with the solid curve of (3)

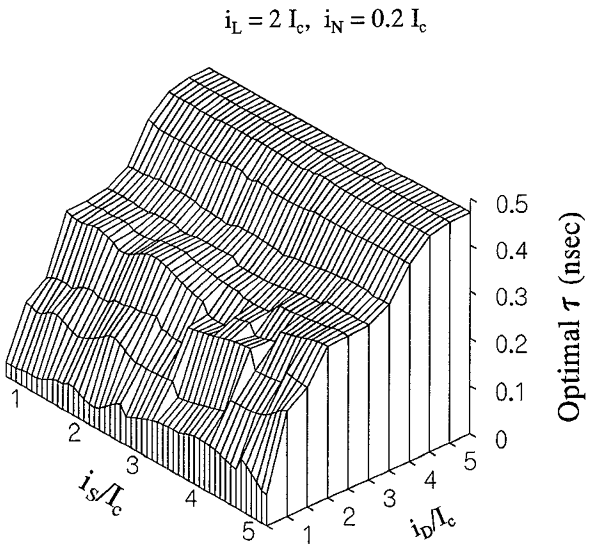

Fig. 4. Optimal τ as a function of I_s and I_p.

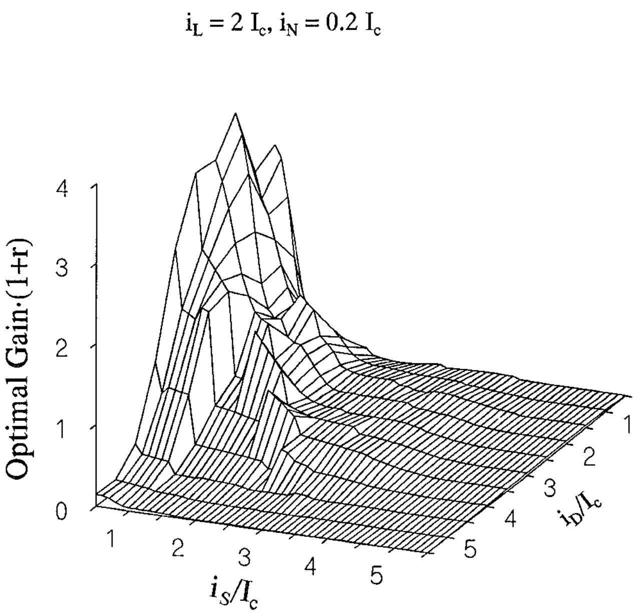

Fig. 5. Optical mixer gain as a function of I_s and I_p.

(without noise). In this example, the (intrinsic) chaotic structure of the calculated gain is by no means random, since $\langle i_N^2 \rangle = 0$. Rather, it corresponds to the rising step edges shown in the solid curve of (3).

Fig. 2(a) shows the Fourier spectrum of $V(t)$ for the following parameters: $i_D = i_S = i_L = 2$, $\langle i_N^2 \rangle = 0.01$, and $\tau = 0.3$ nsec. It is seen in Fig. 2(a) that the spectrum consists of components, which are considerably distinguished from zero, at 0, 1, 9, 10, and 19 GHz. Therefore, the nonlinear frequency mixing occurs at two frequencies, 1 and 19 GHz, which are the difference and the sum of the carrier frequency, 10 GHz, and the local-oscillator frequency, 9 GHz. The gains, multiplied by $(1 + r)$, at 1 and 19 GHz are, respectively, 0.232 and 0.043. In order to distinguish the mixer from a Josephson oscillator in which the oscillation frequency depends only on the dc current drive, i_D, we have plotted in Fig. 2(b)—the Fourier spectrum of $V(t)$ for another set of parameters: $i_D = 3.5$, $i_S = i_L = 2$, $\langle i_N^2 \rangle = 0.01$, and $\tau = 0.3$ nsec. Since only i_D, but not the other parameters, has been changed, one would expect the oscillation frequency to change accordingly if the system describes a Josephson oscillator. However, in Fig. 2(b) mixing frequencies still occur at 1 and 19 GHz, an indication that the junction described by (1) does specify a nonlinear frequency mixer. The gain, multiplied by $(1 + r)$, at 1 and 19 GHz are, respectively, 0.065 and 0.015.

Fig. 3 shows the gain, G, as a function of τ. Here, we have used $i_D = i_S = i_L = 2$, and $\langle i_N^2 \rangle = 0.01$. For large τ values (> 0.2 nsec) the calculated mixer gain approaches a constant. We have found in general that the chaotic motion of the system is most likely to be observed if Shapiro steps are prevalent in the dc response. The gain structure of Fig. 3 is quite universal, and it allows us to estimate the optimal gain occurring at an optimal coupling τ for a given set of circuit parameters, i_D, i_S, i_L, and i_N. Figs. 4 and 5 show such drawings. In Fig. 4, optimal τ is plotted against i_S and i_D, and in Fig. 5 optimal gain, multiplied by $(1 + r)$, is plotted against i_S and i_D. In both figures, $i_L = 2$ and $\langle i_N^2 \rangle = 0.04$ (averaged over ten runs). It is seen in Fig. 5 that for a small value of i_S, the mixer gain can be even larger than unity for suitably chosen biasing conditions. This has been experimentally confirmed by Taur *et al.* [5], indicating the usefulness of a Josephson weak-link junction to be used as a high-gain low-noise broadband mixer.

IV. Conclusion

We conclude that the steady-state solutions of a Josephson circuit can be found by imposing a periodical condition on the system's initial phase. The appearance of intrinsic chaos in a Josephson junction may be associated with Shapiro steps in the dc response. Thus, one should avoid biasing Josephson junctions where there is a Shapiro step, since it represents a potential for chaotic behavior depending on the noise level of the system. Mixers with gain larger than unity can be realized utilizing a Josephson weak-link junction.

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